

SPECTRUM NONINCREASING MAPS ON MATRICES

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ABSTRACT. Maps Φ which do not increase the spectrum on complex matrices in a sense that $\text{Sp}(\Phi(A) - \Phi(B)) \subseteq \text{Sp}(A - B)$ are classified.

1. INTRODUCTION

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices over the complex field \mathbb{C} , and let $\text{Sp}(X)$ be the spectrum of $X \in M_n(\mathbb{C})$. In [9], Marcus and Moyls proved that every linear map $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ preserving eigenvalues (counting multiplicities) is either an isomorphism or an anti-isomorphism. Furthermore, by using their result, one can show that every linear map $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ preserving spectrum of matrices (that is, $\text{Sp}(\Phi(A)) = \text{Sp}(A)$ for all $A \in M_n(\mathbb{C})$) also has the standard form, that is, it is an isomorphism or an anti-isomorphism.

This result has been generalized in different directions. Instead of matrix algebras, the algebras of all bounded linear operators on a complex Banach space were considered, see for example [2, 3, 8, 11] and the references therein. Also, instead of linear or additive preservers, general preservers (without linearity and additivity assumption) of spectrum on $M_n(\mathbb{C})$ were considered. Baribeau and Ransford [4] proved that a spectrum preserving \mathcal{C}^1 diffeomorphism from an open subset of $M_n(\mathbb{C})$ into $M_n(\mathbb{C})$ has the standard form. Mrčun showed in [10] that, if $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a Lipschitz map with $\Phi(0) = 0$ such that $\text{Sp}(\Phi(A) - \Phi(B)) \subseteq \text{Sp}(A - B)$ for all $A, B \in M_n(\mathbb{C})$ then Φ has the standard form. Costara in [7] improved the above result by relaxing Lipschitzian property to continuity. Recently, the continuity of the map was replaced by surjectivity. Namely, in [5], Bendaoud, Douimi and Sarih proved that a surjective map $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfying $\Phi(0) = 0$ and $\text{Sp}(\Phi(A) - \Phi(B)) \subseteq \text{Sp}(A - B)$ for all $A, B \in M_n(\mathbb{C})$ has the standard form. We should mention here that the condition $\Phi(0) = 0$ is a harmless normalization: If Ψ is any map with $\text{Sp}(\Psi(A) - \Psi(B)) \subseteq \text{Sp}(A - B)$, then $\Phi(X) := \Psi(X) - \Psi(0)$ also satisfies this property.

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It is our aim to prove the following generalization of [7, Theorem 1] and [5, Theorem 1.3], in which the maps considered are neither continuous nor surjective.

Theorem 1.1. *Let $n \geq 2$. Suppose that $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a map with $\Phi(0) = 0$ and*

$$\mathrm{Sp}(\Phi(A) - \Phi(B)) \subseteq \mathrm{Sp}(A - B) \quad \text{for all } A, B \in M_n(\mathbb{C}).$$

Then there exists an invertible matrix $S \in M_n(\mathbb{C})$ such that either $\Phi(A) = SAS^{-1}$ for all $A \in M_n(\mathbb{C})$ or $\Phi(A) = SA^tS^{-1}$ for all $A \in M_n(\mathbb{C})$, where A^t denotes the transpose of A .

Remark 1.2. *It was shown by Costara [7] that the maps which satisfy $\mathrm{Sp}(\Phi(A) - \Phi(B)) \supseteq \mathrm{Sp}(A - B)$ are also linear and of a standard form.*

2. STRUCTURAL FEATURES OF BASES OF MATRIX ALGEBRAS

In this section, some features of bases of $M_n(\mathbb{C})$ will be given, which are useful for proving our main result.

Recall that complex numbers $\alpha_1, \dots, \alpha_n$ are linearly independent over \mathbb{Z} , the ring of integers, if the only possibility that $\sum_{i=1}^n z_i \alpha_i = 0$ for some numbers $z_1, \dots, z_n \in \mathbb{Z}$ is $z_1 = z_2 = \dots = z_n = 0$.

Lemma 2.1. *Let $\alpha_1(t), \dots, \alpha_n(t)$ be n linearly independent analytic functions in t , defined in a neighborhood of a closed unit disc Δ . Then the set of all parameters $t_0 \in \Delta$ such that the complex numbers $\alpha_1(t_0), \dots, \alpha_n(t_0)$ are linearly dependent over \mathbb{Z} is at most countable.*

Proof. An analytic function either vanishes identically or it has only finitely many zeros in compact subsets. Since $\alpha_1(t), \dots, \alpha_n(t)$ are linearly independent, none of them can be identically zero, hence each of them has only finitely many zeros in Δ . Let t_1, \dots, t_k be all the zeros of the product $\alpha_1(t)\alpha_2(t)\cdots\alpha_n(t)$ and let $t_0 \in \Delta \setminus \{t_1, \dots, t_k\}$. It is straightforward that there exist integers $z_1, \dots, z_n \in \mathbb{Z}$, with $z_n \neq 0$, such that

$$z_1 \alpha_1(t_0) + \dots + z_n \alpha_n(t_0) = 0$$

if and only if there exist rational numbers $p_i \in \mathbb{Q}$ (in fact, $p_i = \frac{z_i}{z_n}$) such that the meromorphic functions $\widehat{\alpha}_i(t) := \frac{\alpha_i(t)}{\alpha_n(t)}$ satisfy

$$p_1 \widehat{\alpha}_1(t_0) + \dots + p_{n-1} \widehat{\alpha}_{n-1}(t_0) + 1 = 0.$$

By linear independency, the meromorphic function $\alpha_{p_1, \dots, p_{n-1}} : t \mapsto p_1 \widehat{\alpha}_1(t) + \dots + p_{n-1} \widehat{\alpha}_{n-1}(t) + 1$ is nonzero. Thus it has at most finitely many zeros on compact subsets of $\Delta \setminus \{t_1, \dots, t_k\}$, and so it has at most countably many zeros in Δ . Since there are countably many functions $\alpha_{p_1, \dots, p_{n-1}}$ as p_1, \dots, p_{n-1} varies along rational numbers, we get at most $|\mathbb{Q}| \times |\mathbb{Q}| = |\mathbb{Q}|$, i.e., at most countably many points which annihilate one among the functions $\alpha_{p_1, \dots, p_{n-1}}$. Adding also $\{t_1, \dots, t_k\}$, we see that there are at most countably many points

$t \in \Delta$, for which the complex numbers $\alpha_1(t), \dots, \alpha_n(t)$ are linearly dependent with $z_n \neq 0$. If $z_n = 0$ we are seeking for linear dependence of scalars $\alpha_1(t), \dots, \alpha_{n-1}(t)$. By the same argument as before, there are at most countably many such t 's, under additional hypothesis that $z_{n-1} \neq 0$. Proceeding inductively backwards, there are at most countably many parameters $t \in \Delta$ for which $\alpha_1(t), \dots, \alpha_n(t)$ are linearly dependent. \square

To formulate the next technical lemma, we introduce the following notation: given the set $\Omega \subseteq \mathbb{C}$ of cardinality n , let $\vec{\Omega} \subseteq \mathbb{C}^n$ be the set of $n!$ column vectors in \mathbb{C}^n such that the set of their components, relative to a standard basis, equals Ω . For example, if $\Omega = \{1, 2\}$, then $\vec{\Omega} = \{(\frac{1}{2}), (\frac{2}{1})\}$.

Lemma 2.2. *There exist n sets $\Omega_1, \dots, \Omega_n \subseteq \mathbb{C}$, each of cardinality n , such that n vectors x_1, \dots, x_n are linearly independent for any choice of $x_i \in \vec{\Omega}_i$, $i = 1, \dots, n$.*

Proof. It suffices to prove that there exist n column vectors $x_1, \dots, x_n \in \mathbb{C}^n$ such that (i) each of them has pairwise distinct components in a standard basis, and (ii) for arbitrary permutation matrices P_1, \dots, P_n , the vectors $P_1 x_1, \dots, P_n x_n$ are linearly independent. Let x_1 be an arbitrary column vector with pairwise distinct components. Assume $x_1, \dots, x_k, k < n$, are column vectors with the properties from the lemma. Then $\text{Lin}\{P_1 x_1, \dots, P_k x_k\}$ (the linear span of vectors $P_1 x_1, \dots, P_k x_k$) with P_1, \dots, P_k permutation matrices is a k -dimensional linear plane, and there are at most finitely many such distinct k -dimensional subspaces of \mathbb{C}^n for all possible choices of permutation matrices. Since the union of such k -dimensional subspaces is a closed proper subset of the whole space \mathbb{C}^n , there exists a vector $x_{k+1} \in \mathbb{C}^n$ which does not belong to any of these subspaces and has pairwise distinct components. Let P_1, \dots, P_k, P_{k+1} be arbitrary permutation matrices. If $P_{k+1} x_{k+1} \in \text{Lin}\{P_1 x_1, \dots, P_k x_k\}$, then $x_{k+1} \in P_{k+1}^{-1} \text{Lin}\{P_1 x_1, \dots, P_k x_k\} = \{P_{k+1}^{-1} P_1 x_1, \dots, P_{k+1}^{-1} P_k x_k\}$, and since $P_{k+1}^{-1} P_i$, $i = 1, \dots, k$, are again permutation matrices, we obtain a contradiction. Thus $P_1 x_1, \dots, P_{k+1} x_{k+1}$ are linearly independent. By induction on n , the lemma is true. \square

Let \mathcal{K} denotes the set of all matrices in $M_n(\mathbb{C})$ which have n pairwise distinct eigenvalues, linearly independent over \mathbb{Z} . For example, since $\pi = 3^{1/4} \dots$ is not algebraic, the diagonal matrix $\text{diag}(1, \pi, \pi^2, \dots, \pi^{n-1}) \in \mathcal{K}$. We will use the following properties of the set \mathcal{K} .

Lemma 2.3. *Let $B_1, \dots, B_m \in \mathcal{K}$ be a finite sequence of matrices in $M_n(\mathbb{C})$. Then, the set of matrices $X \in \mathcal{K}$ such that $B_k - X \in \mathcal{K}$ for each $k = 1, \dots, m$ is dense in $M_n(\mathbb{C})$.*

Proof. Choose any $A \in M_n(\mathbb{C})$ and let $\varepsilon > 0$. We will show that there exists $X \in \mathcal{K}$ such that $\|A - X\| < \varepsilon$ and $B_k - X \in \mathcal{K}$ for each $k = 1, \dots, m$.

Let $B_0 = 0$. It is easy to see that the set \mathcal{D}_n of matrices with n distinct eigenvalues is dense and open in $M_n(\mathbb{C})$. Hence we may find $A_0 \in \mathcal{D}_n$ arbitrarily close to A . Since \mathcal{D}_n is open, each neighborhood of A_0 , which

is small enough, contains only matrices from \mathcal{D}_n . So we may find $A_1 \in \mathcal{D}_n$ arbitrarily close to A_0 such that $B_1 - A_1 \in \mathcal{D}_n$. Proceeding recursively, there exists $\hat{A} \in M_n(\mathbb{C})$ such that $\|A - \hat{A}\| < \varepsilon$ and that, moreover, $\hat{A} - B_k \in \mathcal{D}_n$ for each $k = 0, \dots, m$. Without loss of generality we write in the sequel A instead of \hat{A} , that is, we assume that $A, A - B_k$ are all in \mathcal{D}_n , $k = 1, \dots, m$.

Note that, for each fixed k , $0 \leq k \leq m$, by Lemma 2.2, there exist n matrices $C_{k,1}, \dots, C_{k,n}$ such that the spectral sets $\text{Sp}(B_k - C_{k,i})$ have the properties stated in Lemma 2.2. Then, for such $C_{k,i}$, it is easily checked that there exists a polynomial $A(x)$ such that $A(0) = A$ and $A((m+n)k+i) = C_{k,i}$ for each $(k, i) \in \mathcal{M} := \{0, \dots, m\} \times \{1, \dots, n\}$.

We claim that, for each fixed k , $0 \leq k \leq m$, $\text{Sp}(B_k - A(x))$ consists of n linearly independent functions which are analytic in a neighborhood of $x = 0$. In fact, the spectrum of analytic perturbation $B_k - A(x)$ of B_k consists of functions which are locally analytic outside a closed discrete set of branching points (see, e.g., [1, Theorem 3.4.25]). Let b_1, \dots, b_r be all the real branching points of modulus smaller or equal to $(m+n)m+n$. Note that for $j \in \{0, (m+n)k+i : (k, i) \in \mathcal{M}\}$, $A(j) = A$ if $j = 0$, and $A(j) = C_{k,i}$ otherwise, and so $\#\text{Sp}(B_k - A(j)) = n$. Hence none of b_i equals some $j \in \{0, (m+n)k+i : (k, i) \in \mathcal{M}\}$. Choose a piecewise linear path $\alpha : [0, (m+n)m+n] \rightarrow \mathbb{C}$ which avoids and does not encircle branching points and passes through $j = (m+n)k+i$, $(k, i) \in \mathcal{M}$. Then the spectral points $\lambda_{k,1}(\alpha(s)), \dots, \lambda_{k,n}(\alpha(s))$ of $\text{Sp}(B_k - A(\alpha(s)))$ are continuous functions of $s \in [0, n+m(m+n)]$. By the construction of $C_{k,i}$, for any choice of $x_1 \in \vec{\text{Sp}}(B_k - C_{k,1}), \dots, x_n \in \vec{\text{Sp}}(B_k - C_{k,n})$, the vectors x_1, \dots, x_n are linearly independent. Then it easily follows that the n functions $\lambda_{k,1}(\alpha(s)), \dots, \lambda_{k,n}(\alpha(s))$ are also linearly independent for each fixed k . Hence, $\lambda_{k,1}(x), \dots, \lambda_{k,n}(x)$ of $\text{Sp}(B_k - A(x))$ are linearly independent analytic functions in a neighborhood of the curve given by path α . Moreover, since $\#\text{Sp}(A(0)) = \#\text{Sp}(A) = n$, $x = 0$ is not a branching point. Thus, also the restrictions of $\lambda_{k,1}(x), \dots, \lambda_{k,n}(x)$ to a neighborhood of $x = 0$ are linearly independent and distinct analytic functions since linear independence is checked by nonvanishing of analytic function, i.e. Wronskian.

Finally, by Lemma 2.1, $B_k - A(x) \in \mathcal{K}$ for every $k \in \{0, \dots, m\}$ and each x outside a countable subset of \mathbb{C} . Hence, there exists x arbitrarily close to 0 such that $B_k - A(x) \in \mathcal{K}$ for every $k \in \{0, \dots, m\}$. Since $A(0) = A$, we can find x such that $X = A(x)$ is close to original matrix A , and that $B_k - X \in \mathcal{K}$ for each k . In particular, $B_0 = 0$ implies that also $X \in \mathcal{K}$. The proof is complete. \square

In particular, Lemma 2.3 implies that the set \mathcal{K} is dense in $M_n(\mathbb{C})$ and hence it contains a basis of $M_n(\mathbb{C})$.

The following proposition is the main result of this section. It gives a interesting structural feature of basis of $M_n(\mathbb{C})$, and is crucial for our proof of Theorem 1.1.

Proposition 2.4. *If $B_1, \dots, B_{n^2} \in \mathcal{K}$ is a basis in $M_n(\mathbb{C})$, then there exists a basis C_1, \dots, C_{n^2} in $M_n(\mathbb{C})$ such that $C_i \in \mathcal{K}$ and $B_i - C_j \in \mathcal{K}$ for every $i, j \in \{1, \dots, n^2\}$.*

Proof. The matrices C_1, \dots, C_{n^2} form a basis if and only if the $n^2 \times n^2$ matrix of their coefficients with respect to the standard basis E_{ij} of $M_n(\mathbb{C})$ (ordered lexicographically) is invertible. Hence, starting with the basis E_{ij} (whose matrix of coefficients is the $n^2 \times n^2$ identity matrix), the small perturbation of E_{ij} is again a basis for $M_n(\mathbb{C})$. We now use the fact that the set

$$\mathcal{K}' = \{C \in \mathcal{K} : C - B_i \in \mathcal{K}, i = 1, 2, \dots, n^2\}$$

is dense in $M_n(\mathbb{C})$ (Lemma 2.3). By the density of \mathcal{K}' , we can find matrices $C_{11}, C_{12}, \dots, C_{nn} \in \mathcal{K}$ with C_{ij} arbitrarily close to E_{ij} , such that $C_{ij} - B_k \in \mathcal{K}$ for each i, j, k . Since they are close to basis E_{ij} , the matrices $C_{11}, C_{12}, \dots, C_{nn}$ are again a basis for $M_n(\mathbb{C})$. \square

3. PROOF OF THEOREM 1.1

In this section we give a proof of our main result. Throughout we always assume that $\Phi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a map satisfying $\Phi(0) = 0$ and $\text{Sp}(\Phi(A) - \Phi(B)) \subseteq \text{Sp}(A - B)$ for all $A, B \in M_n(\mathbb{C})$. First let us state four facts which were already used in a paper by Costara [7]. As usual, $\text{Tr}(X)$ denotes the trace of a matrix X .

Lemma 3.1. *For every $X \in M_n(\mathbb{C})$, we have $\text{Tr}(\Phi(X)) = \text{Tr}(X)$.*

Proof. The proof is the same as the proof of Equation (8) in [7, pp. 2675–2676]. We omit the details. \square

Lemma 3.2. *If $A - B \in \mathcal{K}$, then $\text{Sp}(\Phi(A) - \Phi(B)) = \text{Sp}(A - B)$, counted with multiplicities. In particular, $A \in \mathcal{K}$ implies $\text{Sp } \Phi(A) = \text{Sp } A$.*

Proof. The first claim follows by [7, Lemma 5], Lemma 3.1 and the linearity of trace. The last claim follows by inserting $B = 0$. \square

For any $X \in M_n(\mathbb{C})$, let $S_2(X)$ be the second symmetric function in eigenvalues of a matrix X (i.e., the coefficient of x^{n-2} in characteristic polynomial $p(x) = \det(xI - X)$).

Lemma 3.3. *For every $X \in M_n(\mathbb{C})$, we have $(\text{Tr } X)^2 = \text{Tr}(X^2) + 2S_2(X)$.*

Proof. A straightforward calculation. Also see [7, Eq.(10)]. \square

Lemma 3.4. *If $A, B, (A - B) \in \mathcal{K}$, then $\text{Tr}(AB) = \text{Tr}(\Phi(A)\Phi(B))$.*

Proof. For any $A, B \in \mathcal{K}$ such that $A - B \in \mathcal{K}$, by Lemma 3.3, we have

$$\text{Tr}((A - B)^2) = (\text{Tr}(A - B))^2 - 2S_2(A - B).$$

Since $A - B \in \mathcal{K}$, Lemma 3.2 implies $S_2(A - B) = S_2(\Phi(A) - \Phi(B))$ and $\text{Tr}(A - B) = \text{Tr}(\Phi(A) - \Phi(B))$. It follows that

$$(1) \quad \text{Tr}((A - B)^2) = (\text{Tr}(\Phi(A) - \Phi(B)))^2.$$

Moreover, note that $A, B \in \mathcal{K}$. Likewise, $A, B \in \mathcal{K}$ implies $\text{Tr}(A^2) = \text{Tr}(\Phi(A)^2)$ and $\text{Tr}(B^2) = \text{Tr}(\Phi(B)^2)$. Hence, linearizing (1) gives

$$(2) \quad \text{Tr}(AB) = \text{Tr}(\Phi(A)\Phi(B))$$

whenever $A, B, A - B \in \mathcal{K}$. \square

Proof of Theorem 1.1. Given a matrix $X = (x_{ij}) \in M_n$, let us introduce its row vector $R_X := (x_{11}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{nn})$ and its column vector $C_X := (x_{11}, x_{21}, \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots, x_{nn})^t$. It is elementary that $\text{Tr}(XY) = R_X C_Y$, and hence we may rewrite (2) into

$$(3) \quad R_{\Phi(A)} C_{\Phi(B)} = R_A C_B$$

whenever $A, B, A - B \in \mathcal{K}$ (see also Chan, Li, and Sze [6]).

Now, for any $A \in \mathcal{K}$, by Proposition 2.4, we can find a basis $B_1, \dots, B_{n^2} \in \mathcal{K}$ such that $A - B_i \in \mathcal{K}$. Using B_i in place of B in (3), we obtain a system of n^2 linear equations

$$R_{\Phi(A)} C_{\Phi(B_i)} = R_A C_{B_i}, \quad i = 1, 2, \dots, n^2.$$

Introducing two $n^2 \times n^2$ matrices $\mathcal{U} := [C_{\Phi(B_1)} | C_{\Phi(B_2)} | \dots | C_{\Phi(B_{n^2})}]$ and $\mathcal{C} := [C_{B_1} | \dots | C_{B_{n^2}}]$, this system can be rewritten into

$$(4) \quad R_{\Phi(A)} \mathcal{U} = R_A \mathcal{C}.$$

The identity holds for each $A \in \mathcal{K}$ satisfying $A - B_i \in \mathcal{K}$. By Proposition 2.4 again, there exists another basis $A_1, \dots, A_{n^2} \in \mathcal{K}$ such that $B_i - A_j \in \mathcal{K}$ for every $i, j \in \{1, 2, \dots, n^2\}$. Using A_j in place of A in (4), the identity (4) can be rewritten into a matrix equation

$$\mathcal{V} \mathcal{U} = \mathcal{R} \mathcal{C},$$

where \mathcal{R} is an $n^2 \times n^2$ matrix with j -th row equal to R_{A_j} , and \mathcal{V} is an $n^2 \times n^2$ matrix with j -th row $R_{\Phi(A_j)}$. Since A_1, \dots, A_{n^2} is a basis, the matrix \mathcal{R} is invertible. Likewise, since B_1, \dots, B_{n^2} is a basis, \mathcal{C} is invertible. This implies invertibility of \mathcal{U} . In particular, (4) yields

$$R_{\Phi(A)} = R_A \mathcal{C} \mathcal{U}^{-1} = R_A \mathcal{W}; \quad (\mathcal{W} = \mathcal{C} \mathcal{U}^{-1})$$

for all $A \in \mathcal{K}$ with $A - B_i \in \mathcal{K}$.

Set $\mathcal{K}' = \{A \in \mathcal{K} : A - B_i \in \mathcal{K}, i = 1, 2, \dots, n^2\}$. By Lemma 2.3, the set \mathcal{K}' is dense in $M_n(\mathbb{C})$. Therefore we get

$$R_{\Phi(X)} = R_X \mathcal{W}, \quad \forall X \in \mathcal{K}'.$$

Recall that $\mathcal{W} = \mathcal{C} \mathcal{U}^{-1}$ is an invertible $n^2 \times n^2$ matrix. Now, define a linear bijection $\Psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ by

$$R_{\Psi(X)} = R_X \mathcal{W}, \quad \forall X \in M_n(\mathbb{C}).$$

The map Ψ coincides with Φ on a dense subset \mathcal{K}' . Moreover, Lemma 3.2 implies that $\text{Sp}(\Phi(K)) = \text{Sp}(K)$ for every $K \in \mathcal{K}'$. Hence we have $\text{Sp}(\Psi(K)) = \text{Sp}(K)$ for $K \in \mathcal{K}'$. It follows from the continuity of Ψ and of the spectral function $\text{Sp} : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ that Ψ is a linear bijection satisfying $\text{Sp}(\Psi(X)) =$

$\text{Sp}(X)$ for every $X \in M_n(\mathbb{C})$. Then, by Marcus-Moyls [9], there exists an invertible $S \in M_n(\mathbb{C})$ such that either $\Psi(X) = SXS^{-1}$ for all $X \in M_n(\mathbb{C})$ or $\Psi(X) = SX^tS^{-1}$ for all $X \in M_n(\mathbb{C})$. This gives that either

$$\Phi(K) = SKS^{-1} \quad \text{for all } K \in \mathcal{K}',$$

or

$$\Phi(K) = SK^tS^{-1} \quad \text{for all } K \in \mathcal{K}'.$$

Clearly, neither the hypothesis nor the end result changes if we replace Φ by the map $X \mapsto S^{-1}\Phi(X)S$ or by the map $X \mapsto (S^{-1}\Phi(X)S)^t$. So, with no loss of generality, we can assume that

$$\Phi(K) = K \quad \text{for all } K \in \mathcal{K}'.$$

We assert that $\Phi(X) = X$ for every $X \in M_n(\mathbb{C})$. To show this, write $Y := \Phi(X)$. By the assumption on Φ , for any $K \in \mathcal{K}'$, we have

$$\text{Sp}(K - Y) = \text{Sp}(\Phi(K) - \Phi(X)) \subseteq \text{Sp}(K - X).$$

Since \mathcal{K}' is dense in $M_n(\mathbb{C})$ and as spectral function is continuous, we derive

$$\text{Sp}(A - Y) \subseteq \text{Sp}(A - X) \quad \text{for all } A \in M_n(\mathbb{C}).$$

For any $A \in M_n(\mathbb{C})$, let $B = A - Y$. Then the above relation yields

$$(5) \quad \text{Sp}(B) \subseteq \text{Sp}(B + (Y - X)) \quad \text{for all } B \in M_n(\mathbb{C}).$$

We will show that $Y = X$. Assume on the contrary that $Y - X \neq 0$. In addition, for the sake of convenience, we may also assume that $Y - X$ is already in its Jordan form. If $Y - X$ is not a nilpotent matrix, then inserting $B = -(Y - X)$ in (5) yields a contradiction $\{0\} \neq \text{Sp}(Y - X) \subseteq \text{Sp}(0) = \{0\}$. If $Y - X = J_{n_1} \oplus \cdots \oplus J_{n_k}$ is a nonzero nilpotent matrix in its Jordan form, let $B = (J_{n_1}^{n_1-1} \oplus \cdots \oplus J_{n_k}^{n_k-1})^t$ in (5), where we tacitly assume that $A^0 = I$ for any $A \in M_n(\mathbb{C})$ including the zero matrix. Since $Y - X \neq 0$, there exists at least one block of size ≥ 2 . Clearly then, B is not invertible, but $(Y - X) + B$ is of full rank. This contradicts the fact that $\text{Sp}(B) \subseteq \text{Sp}(B + Y - X)$. Hence $Y = X$, that is, $\Phi(X) = X$ for each $X \in M_n(\mathbb{C})$. \square

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